Combined Conduction and Radiation Heat Transfer in Concentric Cylindrical Media

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The radiative transfer expressions for absorbing, emitting, and nonscattering gray and nongray gases contained between infinitely long concentric cylinders with black surfaces are given in local thermodynamic equilibrium. Resulting energy equations are solved numerically using the undetermined parameters method. A single band, 4.3 μ of CO_2 is considered for the nongray problems. The present solutions for gray and nongray gases obtained in the plane-parallel limit (radius ratio approaches to one) are compared with the plane-parallel results reported in the literature. The present results are obtained for a range of the system parameters, namely, optical thickness, conduction-radiation ratio, and radius ratio. Nongray results in such geometry are obtained for the first time.

Nomenclature

A = total band absorptance, cm^{-1}
A_o = bandwidth parameter, cm^{-1}
$\bar{A}$ = dimensionless total band absorptance, defined as A/A_o
a = constant in exponential-kernal approximation
$B_v(T)$ = Planck function $= C_1/[\exp(C_2/T) - 1]$, W/(cm s)
$B_{vc}(T)$ = Planck function evaluated at band center
B_o^2 = correlation quantity
b = constant in exponential-kernel approximation
C_1 $= 1.1909 \times 10^{12} v^3$, W/(cm s)
C_2 $= 1.4388 v, K$
C_o^2 = correlation quantity
$D_n(x)$ = cylindrical integral function
$D_2(x)$ = cylindrical integral function of the second kind, approximated as in Eq. (16)
E $= \eta(1 - H) + H$
e_1 = total blackbody emissive power $= \sigma T_1^4$
e_2 = total blackbody emissive power $= \sigma T_2^4$
$e_v(T)$ = emissive power evaluated at wave number v and temperature T
$e_{vc}(T)$ = emissive power evaluated at band center and temperature T
$e_{1vc}(T)$ = emissive power evaluated at band center and temperature T_1
$e_{2vc}(T)$ = emissive power evaluated at band center and temperature T_2
H = radius ratio $= r_1/r_2$
I_v = intensity at wave number v
$I_v^+(r)$ = intensity in the direction of increasing r
$I_v^-(r)$ = intensity in the direction of decreasing r
K = thermal conductivity
k_v = absorption coefficient of wave number v, cm^{-1}
L $= (r_2 - r_1)$, cm
M = conduction-radiation ratio defined by Eq. (51)
N = conduction-radiation ratio defined by Eq. (57)
P = total pressure
$q(r)$ = radiative heat flux defined by Eq. (14)
$q(0)$ = radiative heat flux evaluated at the inner cylinder
r = radius
r_1, r_2 = radius of inner and outer cylinder, respectively
r', r'', η' = dummy variables
s = distance along direction of radiative propagation
T = temperature
T_1, T_2 = temperature of inner and outer cylinder, respectively
T_m = mean temperature $= (T_1 + T_2)/2$
t $\equiv \sin\psi$
U = dimensionless coordinate $= C_o^2 Pr$
U_o = dimensionless path length $= C_o^2 P(r_2 - r_1)$
x = variable
τ_o = optical depth $= k_v(r_2 - r_1)$
σ = Stefan-Boltzmann constant 5.668×10^{-12} W/(cm$^2 \cdot K^4$)
η = dimensionless coordinate $= (r - r_1)/(r_2 - r_1)$
α = angle
ψ = angle
v = wave number
vc = band center

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Introduction

RADIATIVE heat transfer combined with conduction and convection appears in many engineering problems associated with combustion, heat exchangers, nuclear reactors, and the cooling of glass.[1–3] Such situations often involve cylindrical geometry. The exact radiative transfer formulation for a gray gas in a cylinder is given by Kuznetsov,[4] Heaslet and Warming,[5] and Kesten.[6] Pigal'skaya[7] was the first to use an exact formulation for the analysis of radiative transfer in a media contained between infinite concentric cylinders. His results are limited, however, to optically thin gray gas cases. Pandey[8] has extended Pigal'skaya's efforts to nongray gases and arbitrary optical thicknesses.

Many approximate methods such as diffusion,[9] differential,[10] variational,[11] modified and higher-order differential,[12–15] and modified moment[16] are applied to determine the solution for gray gas media contained between concentric cylinders under radiative equilibrium. Perlmutter and Howell[17] used a Monte Carlo technique that requires a large amount of computer time to reduce the statistical errors for an acceptable result to the problem. Recently, Pandey and Cogley[18] solved the radiative equilibrium problem for gray and nongray gases contained in concentric cylindrical media. They have studied the effect of radius ratio and optical thickness on the distribution of emissive power and heat flux.

The previous work on the combined conduction and radiation problem is limited to approximate methods for restricted values of the parameters, while the emphasis was on obtaining only the heat flux. Howell[19] used an exchange factor approximation to find the heat flux in concentric cylindrical media. However, the corresponding temperature profile was found for only one case for which the radiative parameters were unclear.

Greif and Clapper[20] found the heat flux in an annular medium by assuming that the conductive and radiative transfer did not interact. Exact solutions developed by Pigal'skaya[7] for the temperature field and heat flux are only valid for an optically thin gray medium. Chang and Smith[21] used the Eddington approximation to solve steady and transient problems. The aim of the present work is to solve the resulting governing equations from the combined conduction and radiation modes of heat transfer for gray and nongray gases contained between infinitely long concentric cylinders for arbitrary optical thickness, conduction-radiation ratio, and radius ratio.

Physical Model and the Radiative Transfer Equation

The coordinate system of the physical model is illustrated in Fig. 1. Both walls are black surfaces maintained at uniform but different temperatures. The medium is emitting and absorbing but nonscattering. Local thermodynamic equilibrium is assumed, and the refractive index is set equal to unity.

The specific intensity I_ν through a participating medium along path s can be expressed as

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = k_\nu(B_\nu - I_\nu) \tag{1}$$

where k_ν is the monochromatic absorption coefficient and B_ν is the Planck function. The derivatives of the intensity along radius r and fixed direction s are related by

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = \frac{\mathrm{d}I_\nu}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}s} \tag{2}$$

Using Fig. 1, the relation between r and s can be written as

$$\frac{\mathrm{d}r}{\mathrm{d}s} = \pm\sqrt{|r^2 - r_2^2 \sin^2\psi|}\,\frac{\cos\alpha}{r} \tag{3}$$

Upon substituting Eqs. (2) and (3) into Eq. (1), one obtains

$$\frac{\mathrm{d}I_\nu^+(r)}{\mathrm{d}r} + F(r,\alpha,\psi)I_\nu^+(r) = F(r,\alpha,\psi)B_\nu(r) \tag{4}$$

and

$$\frac{\mathrm{d}I_\nu^-(r)}{\mathrm{d}r} - F(r,\alpha,\psi)I_\nu^-(r) = -F(r,\alpha,\psi)B_\nu(r) \tag{5}$$

where

$$F(r,\alpha,\psi) = \frac{k_\nu r}{\cos\alpha\sqrt{(r^2 - r_2^2 t^2)}} \tag{6}$$

and

$$t = \sin\psi \tag{7}$$

Here, $I_\nu^+(r)$ represents the intensity in the direction of increasing r, while $I_\nu^-(r)$ is for r decreasing.

The required boundary conditions are

$$I_\nu^+(r_1) = B_\nu(T_1) \tag{8}$$

and

$$I_\nu^-(r_2) = B_\nu(T_2) \tag{9}$$

The solution of Eqs. (4) and (5) with the boundary conditions in Eqs. (8) and (9) can be expressed as

$$I_\nu^+(r) = B_\nu(T_1)\exp\left[-\int_{r_1}^{r} F(r',\alpha,t)\,\mathrm{d}r'\right] + \int_{r_1}^{r} B_\nu(r')F(r',\alpha,t) \times \exp\left[-\int_{r'}^{r} F(r'',\alpha,t)\,\mathrm{d}r''\right]\mathrm{d}r' \tag{10}$$

and

$$I_\nu^-(r) = B_\nu(T_2)\exp\left[-\int_{r}^{r_2} F(r',\alpha,t)\,\mathrm{d}r'\right] - \int_{r_2}^{r} B_\nu(r')F(r',\alpha,t) \times \exp\left[-\int_{r}^{r'} F(r'',\alpha,t)\,\mathrm{d}r''\right]\mathrm{d}r' \tag{11}$$

for

$$0 \le t \le H$$

For $H \le t \le 1$, one obtains

$$\begin{aligned} I_\nu^+(r) = {} & B_\nu(T_2)\exp\left[-\left\{\int_{r_2 t}^{r_2} F(r',\alpha,t)\,\mathrm{d}r' + \int_{r_2 t}^{r} F(r',\alpha,t)\,\mathrm{d}r'\right\}\right] \\ & + \int_{r_2 t}^{r} B_\nu(r')F(r',\alpha,t)\exp\left[-\int_{r'}^{r} F(r'',\alpha,t)\,\mathrm{d}r''\right]\mathrm{d}r' \\ & - \int_{r_2}^{r_2 t} B_\nu(r')F(r',\alpha,t)\exp\left[-\left\{\int_{r_2 t}^{r'} F(r'',\alpha,t)\,\mathrm{d}r'' + \int_{r_2 t}^{r} F(r'',\alpha,t)\,\mathrm{d}r''\right\}\right]\mathrm{d}r' \end{aligned} \tag{12}$$

and

$$I_\nu^-(r) = B_\nu(T_2)\exp\left[-\int_{r}^{r_2} F(r',\alpha,t)\,\mathrm{d}r'\right] - \int_{r_2}^{r} B_\nu(r')F(r',\alpha,t)\exp\left[-\int_{r}^{r'} F(r'',\alpha,t)\,\mathrm{d}r''\right]\mathrm{d}r' \tag{13}$$

The total radiative flux at any radial distance r can be expressed as

$$q(r) = \left(\frac{4r_2}{r}\right)\left[\int_{\Delta\nu}\mathrm{d}\nu\left(\int_0^{\pi/2}\cos^2\alpha\,\mathrm{d}\alpha\left\{\int_0^H [I_\nu^+(r) - I_\nu^-(r)]_I\,\mathrm{d}t + \int_H^{r/r_2} [I_\nu^+(r) - I_\nu^-(r)]_{II}\,\mathrm{d}t\right\}\right)\right] \tag{14}$$

The first term on the right-hand side of Eq. (14) is due to the presence of the inner cylinder, while the other term originates from the outer cylinder (subscripts I and II, respectively). The

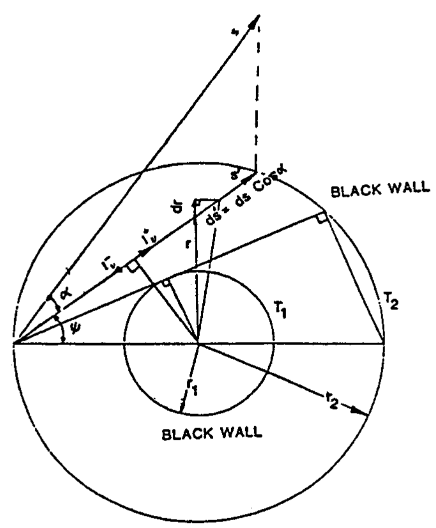

Fig. 1 Concentric cylindrical coordinate system.

expression for the radiative flux can be obtained by substituting Eqs. (10–13) into Eq. (14) as

$$
\begin{aligned}
q(r) = &\left(\frac{4r_2}{r}\right)\Bigg[\int_{\Delta\nu} \mathrm{d}\nu \Bigg(\int_0^H \Bigg\{ B_\nu(T_1) \\
&\times D_3\left[\int_{r_1}^{r} k_\nu(\eta)\eta \,\mathrm{d}\eta \Big/ \sqrt{(\eta^2 - r_2^2 t^2)}\right] \\
&- B_\nu(T_2) D_3\left[\int_{r}^{r_2} k_\nu(\eta)\eta \,\mathrm{d}\eta \Big/ \sqrt{(\eta^2 - r_2^2 t^2)}\right] \\
&+ \int_{r_1}^{r} B_\nu(r')\left[k_\nu(r')r' \Big/ \sqrt{(r'^2 - r_2^2 t^2)}\right] \\
&\times D_2\left[\int_{r'}^{r} k_\nu(\eta)\eta \,\mathrm{d}\eta \Big/ \sqrt{(\eta^2 - r_2^2 t^2)}\right] \mathrm{d}r' \\
&+ \int_{r_2}^{r} B_\nu(r')\left[k_\nu(r')r' \Big/ \sqrt{(r'^2 - r_2^2 t^2)}\right] \\
&\times D_2\left[\int_{r}^{r'} k_\nu(\eta)\eta \,\mathrm{d}\eta \Big/ \sqrt{(\eta^2 - r_2^2 t^2)}\right] \mathrm{d}r' \Bigg\} \mathrm{d}t \\
&+ \int_H^{r/r_2} \Bigg\{ B_\nu(T_2)\Bigg(D_3\Bigg[\int_{r_2 t}^{r_2} k_\nu(\eta)\eta \,\mathrm{d}\eta \Big/ \sqrt{(\eta^2 - r_2^2 t^2)} \\
&+ \int_{r_2 t}^{r} k_\nu(\eta)\eta \,\mathrm{d}\eta \Big/ \sqrt{(\eta^2 - r_2^2 t^2)}\Bigg] \\
&- D_3\left[\int_{r}^{r_2} k_\nu(\eta)\eta \,\mathrm{d}\eta \Big/ \sqrt{(\eta^2 - r_2^2 t^2)}\right]\Bigg) \\
&+ \int_{r_2 t}^{r} B_\nu(r')\left[k_\nu(r')r' \Big/ \sqrt{(r'^2 - r_2^2 r^2)}\right] \\
&\times D_2\left[\int_{r'}^{r} k_\nu(\eta)\eta \,\mathrm{d}\eta / \sqrt{(\eta^2 - r_2^2 t^2)}\right] \mathrm{d}r' \\
&+ \int_{r_2}^{r} B_\nu(r')\left[k_\nu(r')r' \Big/ \sqrt{(r'^2 - r_2^2 t^2)}\right] \\
&\times D_2\left[\int_{r}^{r'} k_\nu(\eta)\eta \,\mathrm{d}\eta \Big/ \sqrt{(\eta^2 - r_2^2 t^2)}\right] \mathrm{d}r' \\
&+ \int_{r_2 t}^{r_2} B_\nu(r')\left[k_\nu(r')r' \Big/ \sqrt{(r'^2 - r_2^2 t^2)}\right] \\
&\times D_2\Bigg[\int_{r_2 t}^{r'} k_\nu(\eta)\eta \,\mathrm{d}\eta \Big/ \sqrt{(\eta^2 - r_2^2 t^2)} \\
&+ \int_{r_2 t}^{r} k_\nu(\eta)\eta \,\mathrm{d}\eta \Big/ \sqrt{(\eta^2 - r_2^2 t^2)}\Bigg] \mathrm{d}r' \Bigg\} \mathrm{d}t \Bigg)\Bigg] \qquad (15)
\end{aligned}
$$

where $D_n(x)$ is the cylindrical integral function defined as

$$
D_n(x) = \int_0^{\pi/2} \cos^{n-1}\alpha \exp\left(\frac{-x}{\cos\alpha}\right) \mathrm{d}\alpha
$$

Equation (15) is an exact formulation for the radiative flux in the concentric cylindrical medium. As a check of its development, Eq. (15) reduces to the equation obtained in Ref. 6 for a cylindrical medium when $H \to 0$ (inner cylinder removed).

The cylindrical integral function $D_n(x)$ can be approximated very accurately by an exponential function[22,23] as

$$
D_2(x) = a \exp(-bx) \qquad (16)
$$

where the following recursion relationship is assumed to hold:

$$
D_n'(x) = -D_{n-1}(x) \qquad n > 1 \qquad (17)
$$

where $D_n'(x)$ is the derivative of $D_n(x)$ with respect to x. These relations reduce Eq. (15) to

$$
\begin{aligned}
q(r) = &\frac{4ar_2}{rb} \\
&\times\Bigg[\int_{\Delta\nu} \mathrm{d}\nu \Bigg\{\int_0^H \Bigg(B_\nu(T_1) \exp\left[-b \int_{r_1}^{r} k_\nu(\eta)\eta \,\mathrm{d}\eta / \sqrt{(\eta^2 - r_2^2 t^2)}\right] \\
&- B_\nu(T_2) \exp\left[-b \int_{r}^{r_2} k_\nu(\eta)\eta \,\mathrm{d}\eta \Big/ \sqrt{(\eta^2 - r_2^2 t^2)}\right] \\
&+ \int_{r_1}^{r} B_\nu(r')[k_\nu(r')r'b / \sqrt{(r'^2 - r_2^2 t^2)}] \\
&\times \exp\left[-b \int_{r'}^{r} k_\nu(\eta)\eta \,\mathrm{d}\eta \Big/ \sqrt{(\eta^2 - r_2^2 t^2)}\right] \mathrm{d}r' \\
&+ \int_{r_2}^{r} B_\nu(r')[k_\nu(r')r'b / \sqrt{(r'^2 - r_2^2 t^2)}] \\
&\times \exp\left[-b \int_{r}^{r'} k_\nu(\eta)\eta \,\mathrm{d}\eta \Big/ \sqrt{(\eta^2 - r_2^2 t^2)}\right] \mathrm{d}r' \Bigg) \mathrm{d}t \\
&+ \int_H^{r/r_2} \Bigg(B_\nu(T_2)\Bigg\{\exp\Bigg[-b\Bigg\{\int_{r_2 t}^{r_2} k_\nu(\eta)\eta \,\mathrm{d}\eta \Big/ \sqrt{(\eta^2 - r_2^2 t^2)} \\
&+ \int_{r_2 t}^{r} k_\nu(\eta)\eta \,\mathrm{d}\eta \Big/ \sqrt{(\eta^2 - r_2^2 t^2)}\Bigg\}\Bigg] \\
&- \exp\left[-b \int_{r}^{r_2} k_\nu(\eta)\eta \,\mathrm{d}\eta \Big/ \sqrt{(\eta^2 - r_2^2 t^2)}\right]\Bigg\} \\
&+ \int_{r_2 t}^{r} B_\nu(r')[k_\nu(r')r'b / \sqrt{(r'^2 - r_2^2 t^2)}] \\
&\times \exp\left[-b \int_{r'}^{r} k_\nu(\eta)\eta \,\mathrm{d}\eta \Big/ \sqrt{(\eta^2 - r_2^2 t^2)}\right] \mathrm{d}r' \\
&+ \int_{r_2}^{r} B_\nu(r')[k_\nu(r')r'b / \sqrt{(r'^2 - r_2^2 t^2)}] \\
&\times \exp\left[-b \int_{r}^{r'} k_\nu(\eta)\eta \,\mathrm{d}\eta \Big/ \sqrt{(\eta^2 - r_2^2 t^2)}\right] \mathrm{d}r' \\
&+ \int_{r_2 t}^{r_2} B_\nu(r')[k_\nu(r')r'b / \sqrt{(r'^2 - r_2^2 t^2)}] \\
&\times \exp\Bigg[-b\Bigg\{\int_{r_2 t}^{r'} k_\nu(\eta) \,\mathrm{d}\eta \Big/ \sqrt{(\eta^2 - r_2^2 t^2)} \\
&+ \int_{r_2 t}^{r} k_\nu(\eta)\eta \,\mathrm{d}\eta \Big/ \sqrt{(\eta^2 - r_2^2 t^2)}\Bigg\}\Bigg] \mathrm{d}r' \Bigg) \mathrm{d}t \Bigg\}\Bigg] \qquad (18)
\end{aligned}
$$

This equation is used in this study.

Radiative Flux for Gray Medium

The gray gas assumption, which states that the absorption coefficient k_ν is independent of frequency (or wave number), can reduce Eq. (18) for the radiative flux in a form similar to that for plane-parallel media[1] as

$$
\begin{aligned}
q(\eta) = &\frac{4a}{\pi bE}\Bigg[H(e_1 - e_2) + \int_0^H \int_0^\eta [e(\eta') - e_1] \\
&\times \{b\tau_o E' / \sqrt{(E'^2 - t^2)}\} \exp[-b\tau_o/(1 - H) \\
&\times \{\sqrt{(E^2 - t^2)} - \sqrt{(E'^2 - t^2)}\}] \,\mathrm{d}\eta' \,\mathrm{d}t \\
&+ \int_0^H \int_1^\eta [e(\eta') - e_2] \\
&\times \{b\tau_o E' / \sqrt{(E'^2 - t^2)}\} \exp[-b\tau_o/(1 - H) \\
&\times \{\sqrt{(E'^2 - t^2)} - \sqrt{(E^2 - t^2)}\}] \,\mathrm{d}\eta' \,\mathrm{d}t
\end{aligned}
$$

$$+\int_H^E \int_{\eta_t}^{\eta} [e(\eta') - e_2]$$
$$\times \{b\tau_o E'/\sqrt{(E'^2 - t^2)}\} \exp[-b\tau_o/(1-H)$$
$$\times \{\sqrt{(E^2 - t^2)} - \sqrt{(E'^2 - t^2)}\}] \, d\eta' \, dt$$
$$+\int_H^E \int_1^{\eta} [e(\eta') - e_2]$$
$$\times \{b\tau_o E'/\sqrt{(E'^2 - t^2)}\} \exp[-b\tau_o/(1-H)$$
$$\times \{\sqrt{(E'^2 - t^2)} - \sqrt{(E^2 - t^2)}\}] \, d\eta' \, dt$$
$$+\int_H^E \int_{\eta_t}^{1} [e(\eta') - e_2]$$
$$\times \{b\tau_o E'/\sqrt{(E'^2 - t^2)}\} \exp[-b\tau_o/(1-H)$$
$$\times \{\sqrt{(E'^2 - t^2)} + \sqrt{(E^2 - t^2)}\}] \, d\eta' \, dt \Big] \quad (19)$$

where

$$\eta = (r - r_1)/(r_2 - r_1) \quad (20)$$

$$\eta_t = (t - H)/(1 - H) \quad (21)$$

$$E = r/r_2 = \eta(1 - H) + H \quad (22)$$

$$\tau_o = k_\nu(r_2 - r_1) \quad (23)$$

$$k_\nu r_2 = \tau_o/(1 - H) \quad (24)$$

$$B_\nu(\eta) = e_\nu(\eta)/\pi \quad (25)$$

$$B_\nu(T_1) = e_{1\nu}/\pi \quad (26)$$

$$B_\nu(T_2) = e_{2\nu}/\pi \quad (27)$$

$$e(\eta) = \int_0^\infty e_\nu(\eta) \, d\eta \quad (28)$$

Note that $e(\eta)$ is the total blackbody emissive power per unit area.

Radiative Flux for Nongray Medium

In the analysis of radiative heat transfer from nongray gases, it is often convenient to use the total or integrated band absorptance A. This quantity is defined as the integral of the spectral absorptivity α_ν over the width of the band $\Delta\nu$ in the following manner:

$$A = \int_{\Delta\nu} \alpha_\nu \, d\nu = \int_{\Delta\nu} \left[1 - \exp\left(k_\nu \frac{Z}{P}\right)\right] d\nu \quad (29)$$

Here, k_ν is the spectral absorption coefficient and Z is the pressure path length that is equal to the product of the radiating gas partial pressure and the physical dimension L. The spectral absorption coefficient k_ν is usually a rapidly varying function of frequency and, therefore, an exact integration of Eq. (29) over a complex vibration-rotation band is not possible in general. A simple continuous expression for the total band absorptance applicable for all path lengths and based on the exponential band model introduced by Tien and Lowder[24] is therefore used. Note that many other exponential band models are reported in the literature. The reason for using the Tien and Lowder[24] correlation was mainly due to the availability of nongray results for plane-parallel media[25] using this model.

The numerical values of the correlation parameters for the gas used in the present work are taken from Refs. 26 and 27. Equation (18) is next modified to include the effects of the band structure of the gas where the absorption coefficient k_ν is assumed to be independent of temperature, i.e.,

$$k_\nu[T,\nu] \equiv k_\nu[T_m,\nu] \quad (30)$$

The Planck function B_ν is also approximated by its value at the center of the band $B_{\nu c}$ for an infrared radiating gas with a sufficiently narrow bandwidth. With these changes, the final expression for the radiative flux for nongray medium can be obtained as

$$q(\eta) = \left(\frac{4a}{\pi b E}\right)\Big[H(e_1 - e_2)$$
$$+\int_0^H [e_{\nu c}(0) - e_{1\nu c}] A_o \bar{A}[bU_o/(1-H)$$
$$\times \{\sqrt{(E^2 - t^2)} - \sqrt{(H^2 - t^2)}\}] \, dt$$
$$-\int_0^E [e_{\nu c}(1) - e_{2\nu c}] A_o \bar{A}[bU_o/(1-H)$$
$$\times \{\sqrt{(1 - t^2)} - \sqrt{(E^2 - t^2)}\}] \, dt$$
$$+\int_H^E [e_{\nu c}(1) - e_{2\nu c}] A_o \bar{A}[bU_o/(1-H)$$
$$\times \{\sqrt{(1 - t^2)} + \sqrt{(E^2 - t^2)}\}] \, dt$$
$$+\int_0^H \int_0^{\eta} \frac{d}{d\eta'} [e_{\nu c}(\nu') - e_{2\nu c}] A_o \bar{A}[bU_o/(1-H)$$
$$\times \{\sqrt{(E^2 - t^2)} - \sqrt{(E'^2 - t^2)}\}] \, d\eta' \, dt$$
$$-\int_0^H \int_1^{\eta} \frac{d}{d\eta'} [e_{\nu c}(\eta') - e_{2\nu c}] A_o \bar{A}[bU_o/(1-H)$$
$$\times \{\sqrt{(E'^2 - t^2)} - \sqrt{(E^2 - t^2)}\}] \, d\eta' \, dt$$
$$+\int_H^E \int_{\eta_t}^{\eta} \frac{d}{d\eta'} [e_{\nu c}(\eta') - e_{2\nu c}] A_o \bar{A}[bU_o/(1-H)$$
$$\times \{\sqrt{(E^2 - t^2)} - \sqrt{(E'^2 - t^2)}\}] \, d\eta' \, dt$$
$$-\int_H^E \int_1^{\eta} \frac{d}{d\eta'} [e_{\nu c}(\eta') - e_{2\nu c}] A_o \bar{A}[bU_o/(1-H)$$
$$\times \{\sqrt{(E'^2 - t^2)} - \sqrt{(E^2 - t^2)}\}] \, d\eta' \, dt$$
$$-\int_H^E \int_{\eta_t}^{1} \frac{d}{d\eta'} [e_{\nu c}(\eta') - e_{2\nu c}] A_o \bar{A}[bU_o/(1-H)$$
$$\times \{\sqrt{(E'^2 - t^2)} + \sqrt{(E^2 - t^2)}\}] \, d\eta' \, dt \Big] \quad (31)$$

where

$$e_{\nu c}(T_1) = e_{1\nu c} \quad (32)$$

$$e_{\nu c}(T_2) = e_{2\nu c} \quad (33)$$

$$U_o = C_o^2 P(r_2 - r_1) \quad (34)$$

and the dimensionless band absorptance is

$$\bar{A} = A/A_o \quad (35)$$

The nonisothermal band absorptance is defined as

$$A[T(y'),L] = \int_{\Delta\nu} \left\{1 - \exp\left[-\int_0^L k_\nu[T(y'),\nu] \, dy'\right]\right\} d\nu \quad (36)$$

which for constant properties reduces the present isothermal band absorptance of

$$A[T_m,L] = \int_{\Delta\nu} \{1 - \exp[-k_\nu(T_m,\nu),L]\} \, d\nu \quad (37)$$

The derivative of this quantity is given by

$$A'[T_m,L] = \int_{\Delta\nu} \{k_\nu(T_m,\nu) \exp[-k_\nu(T_m,\nu),L]\} \, d\nu \quad (38)$$

Governing Energy Equation for Gray Medium

Under steady-state conditions, the energy equation for combined conduction and radiation heat transfer between infinitely long concentric cylinders can be written as

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(Kr\frac{\mathrm{d}T}{\mathrm{d}r}\right)=\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\{rq(r)\} \tag{39}$$

Here, K is the constant thermal conductivity evaluated at the average value of the two wall temperatures, and $q(r)$ is the radiative flux at radius r. The boundary conditions imposed are

$$T=T_1 \quad \text{at} \quad r=r_1 \tag{40}$$

and

$$T=T_2 \quad \text{at} \quad r=r_2 \tag{41}$$

Introducing the dimensionless quantities

$$\theta_g=T/T_2 \tag{42}$$

$$\theta_1=T_1/T_2 \tag{43}$$

into Eq. (39) gives

$$\frac{KT_2}{r_2-r_1}\frac{\mathrm{d}}{\mathrm{d}\eta}\left\{[\eta(1-H)+H]\left(\frac{\mathrm{d}\theta_g}{\mathrm{d}\eta}\right)\right\}=\frac{\mathrm{d}}{\mathrm{d}\eta}\{[\eta(1-H)+H]q(\eta)\} \tag{44}$$

The boundary conditions can now be stated as

$$\theta_g=\theta_1 \quad \text{at} \quad \eta=0 \tag{45}$$

and

$$\theta_g=1 \quad \text{at} \quad \eta=1 \tag{46}$$

Upon integrating Eq. (44) twice with respect to η and using the given boundary conditions, the final expression for the dimensionless temperature distribution $\theta_g(\eta)$ can be expressed in the form

$$\theta_g(\eta)=\theta_{gc}(\eta)-\frac{A_1(\eta)}{N_g}\int_0^1 q(\eta)\,\mathrm{d}\eta+\frac{1}{N_g}\int_0^\eta q(\eta')\,\mathrm{d}\eta' \tag{47}$$

where

$$N_g=\frac{KT_2}{r_2-r_1}=\frac{4e_2N}{\tau_o} \tag{48}$$

$$e_2=\sigma T_2^4 \tag{49}$$

$$A_1(\eta)=1-\left(\frac{\ell nE}{\ell nH}\right) \tag{50}$$

$$N=\frac{Kk_v}{4\sigma T_2^3} \tag{51}$$

$$\theta_{gc}(\eta)=\theta_1+(1-\theta_1)A_1(\eta) \tag{52}$$

The first term $\theta_{gc}(\eta)$ on the right-side of Eq. (47) is the temperature profile due to conduction alone, while the other two terms denote the radiation contribution. The quantities appearing in Eqs. (48–52) are defined in the same way as introduced by Viskanta and Grosh[28] for plane-parallel media to make comparisons easier. The expression for $q(\eta)$ can be obtained from Eq. (19).

Equation (47) is a nonlinear integral equation that cannot be solved analytically. It is therefore solved numerically by using the method of undetermined parameters as described in Ref. 1.

The total heat flux $q_t(0)$ at the inner cylinder ($\eta=0$) can be computed by substituting the obtained temperature profile from Eq. (47) into the expression

$$-\frac{q_t(0)}{e_2}=\frac{4N}{\tau_o}\frac{\partial\theta_g}{\partial\eta}\bigg|_{\eta=0}-\frac{q(0)}{e_2} \tag{53}$$

where

$$\frac{q(0)}{e_2}=\frac{16a}{\pi bH}\left[\int_0^H\int_1^0\theta_g^3\left(\frac{\partial\theta_g}{\partial\eta'}\right)\times\exp\left\{-\frac{b\tau_o}{1-H}[\sqrt{(E'^2-t^2)}-\sqrt{(H^2-t^2)}]\right\}\mathrm{d}\eta'\,\mathrm{d}t\right] \tag{54}$$

The first term appearing on the right side of Eq. (53) is conductive flux evaluated at the inner surface. The second term $q(0)$ is the radiative flux obtained from Eq. (19) at $\eta=0$.

Governing Equation for Nongray Medium

To make the present results comparable with the Chan and Tien[25] solution for plane-parallel media, the dimensionless temperature distribution $\theta(\eta)$ can be expressed from Eq. (39) as

$$\theta(\eta)=A_1(\eta)-\frac{A_1(\eta)}{M}\int_0^1 q^*(\eta)\,\mathrm{d}\eta+\frac{1}{M}\int_0^\eta q^*(\eta')\,\mathrm{d}\eta' \tag{55}$$

where

$$\theta(\eta)=\frac{(T-T_1)}{(T_2-T_1)} \tag{56}$$

$$M=\frac{K}{(r_2-r_1)\dfrac{\partial e_{vc}(T)}{\partial T}\bigg|_{T_m}A_o} \tag{57}$$

$$q^*(\eta)=\frac{q(\eta)}{(T_2-T_1)\dfrac{\partial e_{vc}(T)}{\partial T}\bigg|_{T_m}A_o} \tag{58}$$

and

$$\frac{\partial e_{vc}(T)}{\partial T}\bigg|_{T_m}=\frac{\pi C_1C_2\exp\left(\dfrac{C_2}{T_m}\right)}{\left[T_m\left\{\exp\left(\dfrac{C_2}{T_m}\right)-1\right\}\right]^2} \tag{59}$$

The quantity $A_1(\eta)$ appearing in Eq. (55) is due to conduction alone, while $q^*(\eta)$ is the expression for the dimensionless radiative flux. The expression $q(\eta)$ for nongray gas is given in Eq. (31).

The resulting equations, Eqs. (55) and (31), again are solved numerically by using the method described in Ref. 1. After obtaining the temperature profile from Eq. (55), one computes the total heat flux $q_t(0)$ at the inner cylinder ($\eta=0$) from the equation

$$-\frac{q_t(0)}{e_1}=\frac{A_2}{e_1}\frac{\partial\theta}{\partial\eta}\bigg|_{\eta=0}-\frac{q(0)}{e_1} \tag{60}$$

$$A_2=K\frac{(T_2-T_1)}{(r_2-r_1)} \tag{61}$$

$$q(0)=\frac{4a(e_1-e_2)}{b\pi}-\frac{4a(T_2-T_1)}{b\pi H}\left[\int_0^H\int_1^0\frac{\partial\theta}{\partial\eta'}\frac{\partial e_{vc}(T)}{\partial T}A_o\times\bar{A}[bU_o/(1-H)\{\sqrt{(E'^2-t^2)}-\sqrt{(H^2-t^2)}\}]\,\mathrm{d}\eta'\,\mathrm{d}t\right] \tag{62}$$

The first term on the right side of Eq. (60) is due to conduction alone, evaluated at the inner cylinder ($\eta=0$), while the second term is the contribution due to radiative transfer, computed from Eq. (31) at $\eta = 0$. These numerical results are discussed in the next section.

Gray Results and Discussion

The effects of parameters such as the conduction-radiation ratio N, optical thickness τ_o, and radius ratio H are studied. Numerical values for constants a and b of the exponential-kernel approximation are taken from the work of Habib and Greif.[22] To establish the correctness of the present solution procedure, the results were first obtained in the limit of a plane-parallel medium as a radius ratio H approaches one. Figures 2 and 3 show the effect of radius ratios on the temperature profile for media with two different values of N. One can note from Fig. 2 that the plane-parallel results are in agreement with the present limiting results $H \to 1$. This indicates that the solution procedure is correctly programmed and executing properly. Results for conduction are given in those figures as a reference to show the strength of the radiative transfer. As expected, the radiative effects on the temperature profiles for $N = 0.1$ are small and much less than those for $N = 0.01$ for the optical depth of 0.1. However, these radiative effects become much more pronounced for the larger optical depth of 1.0 used in Fig. 4, with all other parameters kept the same. As has been shown in plane-parallel work,[1] both N and τ_o (namely, the combinations N/τ_o^2 and N for plane-parallel work) control the radiative effects with respect to conduction in this geometry.

Numerical results of the temperature profile and total heat flux for many cases with different orders of polynomials are documented in Pandey.[8] It is important to point out that the satisfactory results for a medium with radius ratios of $H \geq 0.5$ can be obtained by using second/third-order polynomials. Higher-order polynomials would, however, be required for media with smaller radius ratios, e.g., $H = 0.1$.

Figure 5 is another case that was calculated to compare with the results of Chang and Smith[21] based on the Eddington approximation. Here, the parameters are $\tau_o = 1.0$, $N = 0.03$, $\theta_1 = 0.1$, and $H = 0.5$. Though the computational time has become rather large, and third-order polynomial has shown some

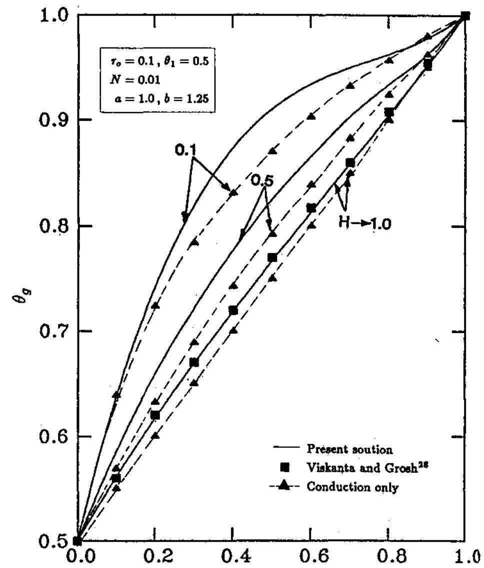

Fig. 2 Effect of radius ratio on temperature profiles of combined conduction and radiation in a concentric cylindrical gray medium.

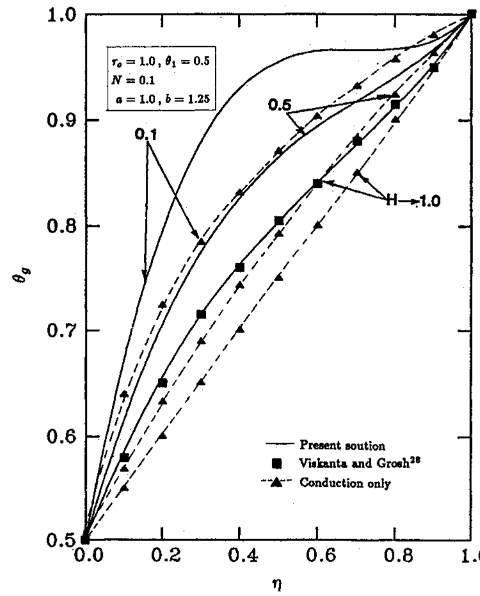

Fig. 4 Effect of radius ratio on temperature profiles of combined conduction and radiation in a concentric cylindrical gray medium.

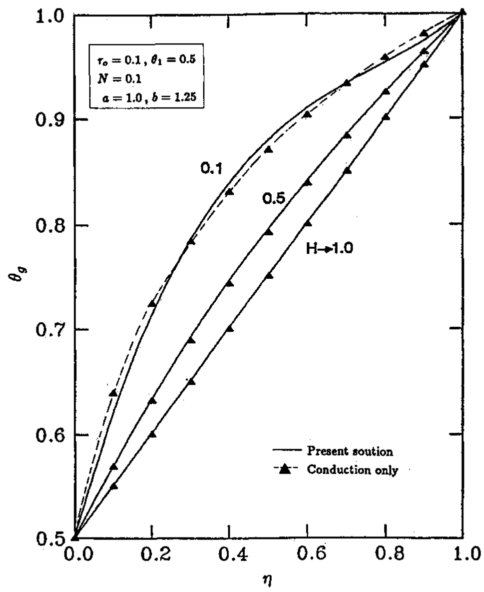

Fig. 3 Effect of radius ratio on temperature profiles of combined conduction and radiation in a concentric cylindrical gray medium.

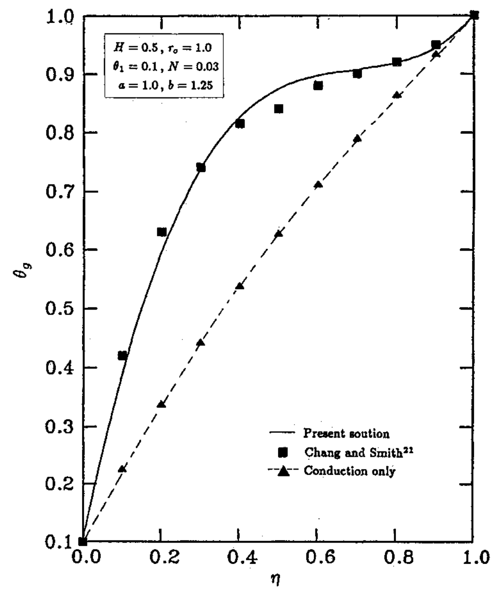

Fig. 5 Temperature profile of combined conduction and radiation in a concentric cylindrical gray medium.

convergence and gives very much the same results as Chang and Smith.[21] Consequently, the Eddington approximation is rather good for this set of parameters. On the other hand, the total heat flux results of Greif and Clapper,[20] assuming no radiation and conduction interaction, show substantial error.

A summary of heat flux results for both previous and present work is presented in Table 1. In the plane-parallel limit, all three approaches show good agreement in the case of $\tau_o = 0.1$ and $N = 0.1$. For cylindrical media, the approach of Greif and Clapper[20] is more accurate at $H = 0.5$ than it is at $H = 0.1$ when compared to the present solution. The present solution, although not exact, due to using lower-order polynomials, is generally better than those found by the approximate method of Greif and Clapper.[20]

Nongray Results and Discussion

Radiative transfer results are not available in the literature for a nongray concentric cylindrical medium. However, plane-parallel results for the 4.3-μ CO_2 band are available from Chan and Tien.[25] This latter work is based on the isothermal band absorptance with a reference temperature equal to the average of the wall temperatures. The radiative parameters considered here are the same as in Ref. 25 so that the present solution procedure can be checked. The important physical parameters of this nongray study are the dimensionless path length U_o, conduction-radiation ratio M, and radius ratio H. Note that the conduction-radiation ratio defined by M is the inverse of that used by Tiwari and Cess.[29] This was done so that M and U_o carry the same meaning as N and τ_o in the gray problems that have already been discussed. Numerical values for contents a and b of the exponential kernel approximation are taken from Ref. 23.

As in the previous nongray work, two problems are solved where the radiative parameters are chosen to match those in Ref. 25. These problems are called cases 1 and 2 and differ in that the dimensionless path length U_o is 40 times larger in case 2 than in case 1. Figure 6 gives the temperature distribution in the medium for different radius ratios H when the conduction mode is dominating ($M = 3.346$). One can observe that the present solution obtained in the plane-parallel limit ($H \rightarrow 1$) is in close agreement with the results obtained by Chan and Tien.[25] This establishes that our solution procedure is correct in the plane-parallel limit and consequently gives us confidence that the cylindrical results are also accurate. Note that the computing times for combined radiation and conduction are rather large; this limits the order of the polynomial that can be considered. The results for radius ratios of 1.0 and 0.5 are almost asymptotic for third-order polynomials. However, this is not true for $H = 0.1$, since the solutions change rather strongly going from a second- to third-order polynomial. Higher-order polynomials for this radius ratio were not considered because computational costs become prohibitive.

Figure 7 shows the temperature distributions for different radius ratios H in a medium where the radiation is 12 times larger than the conduction mode ($M = 0.0836$). Again, the present solution in the plane-parallel limit for a third-order of

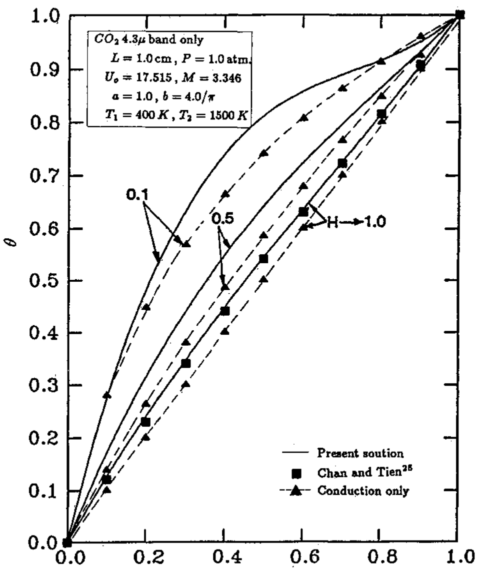

Fig. 6 Effect of radius ratio on temperature profiles of combined conduction and radiation in a concentric cylindrical nongray medium (case 1).

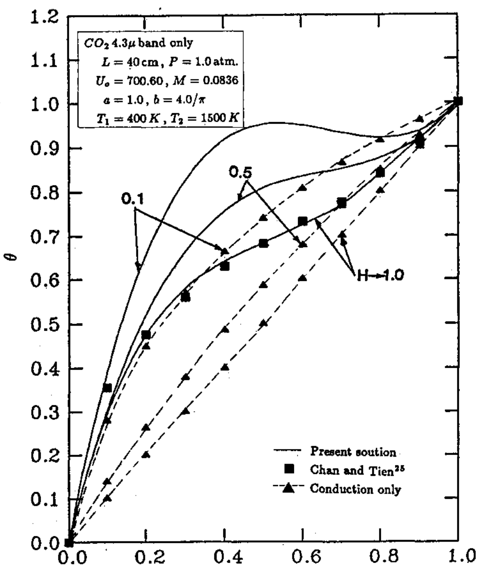

Fig. 7 Effect of radius ratio on temperature profiles of combined conduction and radiation in a concentric cylindrical nongray medium (case 2).

Table 1 Comparison of dimensionless total heat flux = $q_r(0)/e_2$ evaluated at inner cylinder ($\eta = 0$) due to combined conduction and radiation in gray media[a]

		$H \rightarrow 1.0$			$H = 0.5$		$H = 0.1$	
τ_o	N	Viskanta and Grosh[28]	Greif and Clapper[b]	Present work	Greif and Clapper[b]	Present work	Greif and Clapper[b]	Present work
0.1	0.1	2.877	2.872	2.885	1.888	1.947	0.874	1.324
0.1	0.01	1.076	1.072	1.090	0.590	0.628	0.170	0.239
1.0	0.1	0.799	0.736	0.739	0.453	0.540	0.157	0.307
1.0	0.01	0.594	0.556	0.546	0.323	0.381	0.086	0.158

[a]Present solution was obtained by using a polynomial of order 2 and the exponential-kernel constants. $a = 1.0$ and $b = 1.25$ for different optical thicknesses τ_o and conduction-radiation ratios N.
[b]Computed from Eq. (6) of Ref. 20.

polynomial agrees with the equivalent results by Chan and Tien.[25] Numerical results used in these figures along with the dimensionless total heat flux $-q_t(0)/e_1$ and computing time (CPU) are documented in Pandey.[8] One should note here that the third-order polynomials require large computational times and that for the present smaller value of M, none of the solutions can be considered asymptotic. There is little doubt that the present solution procedure will converge to a correct solution if unlimited computer time is given. It is unfortunately true that radiative transfer problems like the present one require large computational times, but such results can be used as benchmark solutions to develop approximate methods in the future.

Conclusions

The problem of combined conduction and radiation heat transfer in absorbing, emitting, and nonscattering gray and nongray gases contained between infinitely long concentric cylinders with black surfaces was solved numerically. A single band of CO_2 was considered for the nongray problems. The results for gray and nongray gases were obtained for a range of system parameters, namely, optical thickness, conduction-radiation ratio, and radius ratio.

Present results for gray and nongray gases obtained in the plane-parallel limit (radius ratio approaches one) were found in close agreement with the other plane-parallel medium results reported in the literature. Satisfactory results in some cases can be achieved by using second/third-order polynomials. The determination of asymptotic results for a medium with radius ratio of 0.1 would, however, require higher-order polynomials, which may become computationally expensive. Nongray results for cylindrical geometries were obtained for the first time.

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